$i=1 \quad \partial \dot{q}_i \quad \dot{q}_i = T_1$

P.G. Sem-III
Paper-XIII, unit-I
classical mechanics

2005, 07

Energy integral or integral of energy or energy equation : (OR Deduce the principle of energy from Lagrange's eqn)

The general Lagrange's equation of motion may be written as $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} = \tilde{Q}_i$

where Lagrangian function $L = T - V$,

$T = T(q_i, \dot{q}_i, t)$, $V = V(q_i, t)$

and $\tilde{Q}_i$ = Non-potential force

$= \tilde{Q}_i(q_i, \dot{q}_i, t)$

Multiplying the equation by $\dot{q}_i$ and adding from $i = 1$ to n, we get –

$$\sum_{i=1}^{n} \dot{q}_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right] = \sum_{i=1}^{n} \tilde{Q}_i \cdot \dot{q}_i \quad \text{— (1)}$$

Now we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \cdot \dot{q}_i\right) = \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$$

$$\Rightarrow \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right) - \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i$$

with the help of this (1) reduces to

$$\sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right) - \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i - \frac{\partial L}{\partial q_i} \dot{q}_i\right] = \sum_{i=1}^{n} \tilde{Q}_i \dot{q}_i \quad \text{— (2)}$$

Again $L = L(q_i, \dot{q}_i, t)$

$$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial L}{\partial t}$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i = \frac{dL}{dt} - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \dot{q}_i - \frac{\partial L}{\partial t}$$

with the help of this (2) reduces to

$$\sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right) - \frac{dL}{dt} + \cancel{\frac{\partial L}{\partial q_i} \dot{q}_i} + \frac{\partial L}{\partial t} - \cancel{\frac{\partial L}{\partial q_i} \dot{q}_i}\right] = \sum_{i=1}^{n} \tilde{Q}_i \dot{q}_i$$

$$\Rightarrow \frac{d}{dt}\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - \frac{dL}{dt} = \sum_{i=1}^{n}\tilde{Q}_i\dot{q}_i - \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt}\sum_{i=1}^{n}\frac{\partial}{\partial \dot{q}_i}(T_2+T_1+T_0-V)\dot{q}_i - \frac{d}{dt}(T_2+T_1+T_0-V)$$
$$= \sum_{i=1}^{n}\tilde{Q}_i\dot{q}_i - \frac{\partial}{\partial t}(T_2+T_1+T_0-V)$$

$$\Rightarrow \frac{d}{dt}(2T_2+T_1) - \frac{d}{dt}(T_2+T_1+T_0-V)$$
$$= \sum_{i=1}^{n}\tilde{Q}_i\dot{q}_i - \frac{\partial T}{\partial t} + \frac{\partial V}{\partial t}$$

as T_2 and T_1 are homogeneous function of second and first degree in generalised velocities respectively.

$$\Rightarrow \frac{d}{dt}(T_2+T_1) - \frac{d}{dt}(T_1+T_0-V)$$
$$= \sum_{i=1}^{n}\tilde{Q}_i\dot{q}_i - \frac{\partial T}{\partial t} + \frac{\partial V}{\partial t}$$

$$\Rightarrow \frac{d}{dt}(T_2+T_1+T_0+V) - \frac{d}{dt}(T_1+2T_0)$$
$$= \sum_{i=1}^{n}\tilde{Q}_i\dot{q}_i - \frac{\partial T}{\partial t} + \frac{\partial V}{\partial t}$$

$$\Rightarrow \frac{d}{dt}(T+V) - \frac{d}{dt}(T_1+2T_0) = \sum_{i=1}^{n}\tilde{Q}_i\dot{q}_i - \frac{\partial T}{\partial t} + \frac{\partial V}{\partial t}$$

Integrating both sides, we get—

$$T+V = T_1 + 2T_0 + \int\left[\sum_{i=1}^{n}\tilde{Q}_i\cdot\dot{q}_i - \frac{\partial T}{\partial t} + \frac{\partial V}{\partial t}\right]dt + h$$

where h is constant.

This is general energy integral or energy equation or integral of energy.

Case 1. If the system is scleronomic

In this Case $T_1 = T_0 = \frac{\partial T}{\partial t} = 0$

In this Case energy equation reduces to

$$T+V = \int \left[\sum_{i=1}^{n} \tilde{Q}_i \cdot \dot{q}_i + \frac{\partial V}{\partial t} \right] dt + h$$

Case 2. when the system is scleronomic and the potential energy does not contain t explicitly

Then energy integral may be written as

$$T+V = \int \sum_{i=1}^{n} \tilde{Q}_i \, \dot{q}_i \, dt + h$$

Case 3. when the system is conservative then

(i) The system is scleronomic

(ii) potential energy V does not contain t explicitly.

(iii) Non-potential forces does not exist.

Then in this Case energy integral or energy equation may be written as

$$T+V = h \text{ (Constant)}.$$